

STILL IN LIGHT-CONE SUPERSPACE

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Abstract

The recently formulated Bagger-Lambert-Gustavsson (BLG) theory in three dimensions is described in terms of a constrained chiral superfield in light-cone superspace. We discuss the use of Superconformal symmetry to determine the form of its interactions, in complete analogy with $N = 4$ SuperYang-Mills in four dimensions.

1 Introduction

Maximally supersymmetric theories live in two different superspaces. The first with eight complex Grassmann variables is used to describe $N = 1$ supergravity in $d = 11$, $N = 8$ supergravity in $d = 4$, $N = 16$ Supergravity in $d = 3$, and so-on. With a dimensionful coupling, these theories are not superconformal. Instead they contain non-compact and non-linear symmetries, $E_{7(7)}$ and $E_{8(8)}$ in $d = 4$ and 3 , respectively.

The second superspace with only four complex Grassmann variables is equally rich. It houses theories with Superconformal symmetry in $d = 6, 5, 4$ and 3 dimensions. The latter theory has been recently formulated covariantly [1, 2], and on the light-cone [3].

It has already been shown[4] how the fully interacting $N = 4$ SuperYang-Mills theory[5] in $d = 4$ can be determined by requiring $PSU(2, 2|4)$ Superconformal symmetry on a constrained chiral superfield in light-cone superspace with four complex Grassmann variables.

The following is a progress report on using the same technique, now applied to $OSp(2, 2|8)$ Superconformal symmetry on the same chiral superfield. On the light-cone, supersymmetries split into kinematical and dynamical supersymmetries. Kinematical supersymmetries are linearly realized on the chiral superfield. The dynamical ones also contain a linear term (free theory), but also terms non-linear in the (super)fields, which, in superconformal theories, suffice to *completely* determine the theory. Our technique has been to use algebraic consistency to find its expression.

Consistency with the kinematical constraints yields two possible expressions for the dynamical supersymmetries, each determined in terms of four integers, and with an unknown four-index tensor f^{abcd} , where the indices label the superfields. The values of these integers are determined by requiring that the

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light-cone Hamiltonian and boosts commute with one another. At Shifman (this proceeding), we reported an unexpected solution, with fractional light-cone derivatives acting on the chiral superfield, without assuming any symmetry among the indices of f^{abcd} . Since then, we have found [6] that by requiring antisymmetry in three of its indices, $b \leftrightarrow c$, $b \leftrightarrow d$ and $c \leftrightarrow d$, the BLG solution emerges from these algebraic constraints, apparently uniquely.

2 Superconformal Theories

In 1978, W. Nahm [7] catalogued all relativistic field theories which extended the Poincaré symmetry to Superconformal symmetry. We only list those in spacetime dimensions $d = 6, 5, 4$, and 3 :

$$\begin{aligned} d = 6 & \quad OSp(2n | 6, 2) \supset SO(6, 2) \times Sp(2n)_R , \\ d = 5 & \quad F[4] \supset SO(5, 2) \times SU(2)_R , \\ d = 4 & \quad SSU(2, 2 | n) \supset SO(4, 2) \times SU(n)_R \times U(1)_R , \\ d = 4 & \quad PSU(2, 2 | 4) \supset SO(4, 2) \times SU(4)_R , \\ d = 3 & \quad OSp(2, 2 | n) \supset SO(3, 2) \times SO(n)_R , \end{aligned}$$

using Kac's notation for the superalgebras. The conformal group in d spacetime dimensions is $SO(d, 2)$; for $d = 4$, it is a non-compact form of $SU(2, 2)$, and for $d = 3$ it is isomorphic to $Sp(2, 2)$. These theories have large global R -symmetries. The theories with special number of R -symmetries, $n = 2$ in $d = 6$, $n = 4$ in $d = 4$, and $n = 8$ in $d = 3$, can be described in terms of constrained chiral superfields in light-cone superspace. Since then, it has been realized that many superconformal theories are seminal, not only in quantum field theory but also in Superstrings and M-theory [8].

3 $N = 4$ Light-Cone Superspace

We introduce the usual light-cone variables

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3) , \quad \partial^\pm = \frac{1}{\sqrt{2}}(\partial^0 \pm \partial^3) , \quad (1)$$

and denote the transverse variables by $x_1 \dots x_{d-2}$. The relevant superspace contains four complex Grassmann variables, θ^m and $\bar{\theta}_m$, in terms of which we define the chiral derivatives

$$d^m = -\frac{\partial}{\partial \bar{\theta}_m} - \frac{i}{\sqrt{2}} \theta^m \partial^+ ; \quad \bar{d}_n = \frac{\partial}{\partial \theta_n} + \frac{i}{\sqrt{2}} \bar{\theta}_n \partial^+ ;$$

they satisfy

$$\{d^m, \bar{d}_n\} = -i\sqrt{2}\delta_n^m \partial^+ . \quad (2)$$

The chiral superfields

$$\begin{aligned} \varphi^a(y) = & \frac{1}{\partial^+} A^a(y) + \frac{i}{\sqrt{2}} \theta^m \theta^n \bar{C}_{mn}^a(y) + \frac{1}{12} \theta^m \theta^n \theta^p \theta^q \epsilon_{mnpq} \partial^+ \bar{A}^a(y) \\ & + \frac{i}{\partial^+} \theta^m \bar{\chi}_m^a(y) + \frac{\sqrt{2}}{6} \theta^m \theta^n \theta^p \epsilon_{mnpq} \chi^{qa}(y) \end{aligned}$$

where a is a taxonomic index, are chiral by construction,

$$d^m \varphi^a(y) = 0 , \quad (3)$$

where the component fields depend on the chiral coordinates

$$y = (x_1, \dots, x_{d-2}, x^- - i \frac{\theta^m \bar{\theta}_m}{\sqrt{2}}) . \quad (4)$$

The parameter x^+ is set to zero without loss of generality. The chiral superfields obey the “inside-out” constraint

$$\bar{d}_m \bar{d}_n \varphi^a = \frac{1}{2} \epsilon_{mnpq} d^p d^q \bar{\varphi}^a . \quad (5)$$

In $d = 4$ SuperYang-Mills, this important constraint allowed us[4] to write its light-cone interacting Hamiltonian as a *positive definite* quadratic form.

The component fields of each chiral superfield represent sixteen physical degrees of freedom, eight bosons and eight fermions. They are organized in terms of an $SO(8)$ R -symmetry, with the bosons transforming as a vector, the fermions as a spinor.

Introduce the operators

$$q^m = -\frac{\partial}{\partial \theta_m} + \frac{i}{\sqrt{2}} \theta^m \partial^+ ; \quad \bar{q}_n = \frac{\partial}{\partial \theta^n} - \frac{i}{\sqrt{2}} \bar{\theta}_n \partial^+ , \quad (6)$$

which satisfy

$$\{q^m, \bar{q}_n\} = i\sqrt{2}\delta_n^m \partial^+ , \quad (7)$$

and do not alter chirality, since they anticommute with the chiral derivatives

$$\{q^m, \bar{d}_n\} = \{q^m, d^n\} = 0 . \quad (8)$$

The $SO(8)$ transformations are written in terms of those of its $SO(6) \times U(1) \sim SU(4) \times U(1)$ subgroup, with parameters ω_n^m , and ω :

$$\delta_{SO(6)} \varphi^a = \omega_n^m \frac{i}{\sqrt{2}} \left(q^n \bar{q}_m - \frac{1}{4} \delta_m^n q^l \bar{q}_l \right) \frac{1}{\partial^+} \varphi^a ; \quad (9)$$

$$\delta_{U(1)} \varphi^a = \omega \frac{i}{4\sqrt{2}} (q^m \bar{q}_m - \bar{q}_m q^m) \frac{1}{\partial^+} \varphi^a ; \quad (10)$$

and the coset parameters ω^{mn} , and $\bar{\omega}_{mn}$,

$$\delta_{\text{coset}} \varphi^a = \omega^{mn} \frac{i}{\sqrt{2}} \bar{q}_m \bar{q}_n \frac{1}{\partial^+} \varphi^a ; \quad \delta_{\text{coset}} \varphi^a = \omega_{mn} \frac{i}{\sqrt{2}} q^m q^n \frac{1}{\partial^+} \varphi^a , \quad (11)$$

This chiral superfield can be used to define theories in different dimensions, with the only modifications of increase the number of transverse coordinates of its component fields:

- $d = 10$

The superfield describes $N = 1$ in $d = 10$ dimensions. This theory is not superconformal, as it is the zero slope limit of an open superstring. The $SO(8)$ transformations are interpreted as the “spin” part of the transverse little group, the orbital part being supplied by the appropriate number of transverse coordinates. There are no modifications to the chiral superfield, except for the dependence of its components on the six extra transverse coordinates.

- $d = 6$

The superconformal group is $OSp(4|6,2)$. The transverse light-cone little group is $SO(4) \sim SU(2) \times SU(2)$. The first $SU(2)$ has only an orbital part, whereas the spin part of the second $SU(2)$ is to be found in the decomposition

$$SO(8) \supset SU(2) \times Sp(4)_R , \quad (12)$$

where the physical fields decompose as

$$\mathbf{8}_b = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{5}) , \quad \mathbf{8}_f = (\mathbf{2}, \mathbf{4}) . \quad (13)$$

The bosons split into an R -quintet of scalar fields and an R -singlet tensor, a second rank antisymmetric tensor with self-dual three-form field strength.

- $d = 5$

The superconformal symmetry group is $F[4]$. The transverse little group is $SO(3) \sim SU(2)$, and its spin part is to be found in the decomposition

$$SO(8) \supset SU(2) \times SU(2)_R . \quad (14)$$

The R -symmetry reduces to $SU(2)$. This decomposition is similar to that in $d = 6$, with the anomalous embedding of $SU(2)$ in $Sp(4)$ with

$$Sp(4) \supset SU(2) , \quad \mathbf{5} = \mathbf{5} , \quad \mathbf{4} = \mathbf{4} , \quad (15)$$

so that the scalar bosons split into one R -singlet vector, and five scalars with R -spin 2 and the fermions R -spin 3/2.

- $d = 4$

The little group is now just $SO(2)$ whose spin part is found in

$$SO(8) \supset SO(2) \times SO(6)_R \sim U(1) \times SU(4)_R . \quad (16)$$

This leads to the well-known $N = 4$ SuperYang-Mills theory, symmetric under $PSU(2, 2|4)$, with one vector and six scalars.

- $d = 3$

There is no light-cone little group, and the R -symmetry is the full $SO(8)$. The chiral superfield describes the degrees of freedom in the Nahm theory with $n = 8$, and symmetry $OSp(2, 2|8)$. The bosons (fermions) form an R -symmetry vector (spinor) octet.

The light-cone formulation of this theory will occupy the rest of this paper, using algebraic techniques previously developed for the $N = 4$ theory in four dimensions.

4 $OSp(2, 2|8)$ Generators

We begin with

$$OSp(2, 2|8) \supset Sp(2, 2) \times SO(8) ,$$

where the first factor group is the conformal group in three dimensions, and the second factor group is the R -symmetry.

In light-cone coordinates, the space-time generators are either kinematical or dynamical. The kinematical generators operate within the initial surface ($x^+ = 0$), while the dynamical generators, called hamiltonians by Dirac, act transversely to the initial surface. The kinematical operators are the same in free and interacting theories, and are linear in the (super)fields. The dynamical operators also contain a part linear in the (super)fields for the free theory, but in the interacting theory, they develop non-linear dependence on the (super)fields.

The ten generators of the conformal group in three dimensions are given by

$$\text{Conformal Group} \quad \left\{ \begin{array}{l} \text{Lorentz Group : } J^{+-}, J^+ ; \quad \mathcal{J}^- \\ \text{Translations : } P, P^+ ; \quad \mathcal{P}^- \\ \text{Dilatation : } D \\ \text{Conformal : } K, K^+ ; \quad \mathcal{K}^- \end{array} \right. ,$$

with the dynamical generators written in capital calligraphic letters. Note that J^+ and K^+ can be viewed as kinematical as long as we set the parameter x^+ to zero.

The supersymmetry and superconformal generators

$$\text{Supers} \quad \left\{ \begin{array}{l} \text{Supersymmetry : } q, \bar{q} ; \quad \mathcal{Q}, \bar{\mathcal{Q}} \\ \text{Superconformal : } s, \bar{s} ; \quad \mathcal{S}, \bar{\mathcal{S}} \end{array} \right. ,$$

also split into kinematical and dynamical operators. All R -symmetry generators are kinematical, and given by Eqs.(9-11).

4.1 Kinematical Transformations

They are expressed in terms of

$$\mathcal{N} = \theta^m \frac{\partial}{\partial \theta^m} + \bar{\theta}_m \frac{\partial}{\partial \bar{\theta}_m} ; \quad \mathcal{A} \equiv x^- \partial^+ - \frac{x}{2} \partial - \frac{1}{2} \mathcal{N} + \frac{1}{2} . \quad (17)$$

The kinematical Poincaré transformations are

$$\delta_{P+} \varphi^a = -i \partial^+ \varphi^a ; \quad \delta_P \varphi^a = -i \partial \varphi^a ; \quad (18)$$

$$\delta_{J+} \varphi^a = i x \partial^+ \varphi^a ; \quad \delta_{J+-} \varphi^a = i (x^- \partial^+ - \frac{1}{2} \mathcal{N} + 1) \varphi^a ; \quad (19)$$

$$\delta_{P+} \varphi^a = -i \partial^+ \varphi^a ; \quad \delta_P \varphi^a = -i \partial \varphi^a , \quad (20)$$

followed by the kinematical conformal symmetries

$$\delta_D \varphi^a = i (x^- \partial^+ - x \partial - \frac{1}{2} \mathcal{N} + \frac{1}{2}) \varphi^a ; \quad (21)$$

$$\delta_K \varphi^a = 2i x \mathcal{A} \varphi^a ; \quad \delta_{K+} \varphi^a = i x^2 \partial^+ \varphi^a . \quad (22)$$

Similarly, the kinematical (spectrum generating) supersymmetries, with parameters ε^m and $\bar{\varepsilon}_m$, are

$$\delta_{\varepsilon \bar{q}}^{kin} \varphi^a = \varepsilon^m \bar{q}_m \varphi^a ; \quad \delta_{\bar{\varepsilon} q}^{kin} \varphi^a = \bar{\varepsilon}_m q^m \varphi^a , \quad (23)$$

and finally kinematical superconformal transformations with parameters α^m and $\bar{\alpha}_m$

$$\delta_{\alpha \bar{s}} \varphi^a = -i x \alpha^m \bar{q}_m \varphi^a ; \quad \delta_{\bar{\alpha} s} \varphi^a = i x \bar{\alpha}_m q^m \varphi^a . \quad (24)$$

4.2 Free Dynamical Transformations

In superconformal theories, **all** dynamical generators are determined by the algebra from the dynamical supersymmetry transformations, because the algebra is simple. To see how this works, we start from the *free* dynamical supersymmetry transformations (written in bold), which are given by

$$\delta_{\varepsilon\overline{Q}}^{free} \varphi^a = \frac{1}{\sqrt{2}} \varepsilon^m \bar{q}_m \frac{\partial}{\partial^+} \varphi^a, \quad \delta_{\varepsilon Q}^{free} \varphi^a = \frac{1}{\sqrt{2}} \bar{\varepsilon}_m q^m \frac{\partial}{\partial^+} \varphi^a. \quad (25)$$

We then use the commutators

$$[\delta_{\varepsilon\overline{Q}}, \delta_{\varepsilon Q}] \varphi^a = \sqrt{2} \bar{\varepsilon}_m \varepsilon^m \delta_{P^-} \varphi^a \rightarrow \delta_{P^-} \varphi^a, \quad (26)$$

$$[\delta_K, \delta_{P^-}] \varphi^a = 2i \delta_{J^-} \varphi^a \rightarrow \delta_{J^-} \varphi^a, \quad (27)$$

$$[\delta_K, \delta_{J^-}] \varphi^a = -i \delta_{K^-} \varphi^a \rightarrow \delta_{K^-} \varphi^a, \quad (28)$$

$$[\delta_K, \delta_{\varepsilon\overline{Q}}] \varphi^a = \sqrt{2} \delta_{\varepsilon\overline{S}} \varphi^a \rightarrow \delta_{\varepsilon\overline{S}} \varphi^a, \quad (29)$$

to compute the remaining dynamical transformations. Evaluation of the commutators yields

$$\begin{aligned} \text{Time Translation :} \quad & \delta_{P^-}^{free} \varphi^a = -i \frac{\partial^2}{2 \partial^+} \varphi^a, \\ \text{Lorentz Boost :} \quad & \delta_{J^-}^{free} \varphi^a = -i \frac{\partial}{\partial^+} \mathcal{A} \varphi^a, \\ \text{Conformal Boost :} \quad & \delta_{K^-}^{free} \varphi^a = 2i \frac{1}{\partial^+} \mathcal{A} (\mathcal{A} - \frac{1}{2}) \varphi^a, \\ \text{SuperConformal :} \quad & \delta_{\alpha\overline{S}}^{free} \varphi^a = i \alpha^m \bar{q}_m \frac{1}{\partial^+} \mathcal{A} \varphi^a, \\ & \delta_{\alpha S}^{free} \varphi^a = -i \bar{\alpha}_m q^m \frac{1}{\partial^+} \mathcal{A} \varphi^a. \end{aligned}$$

These are valid in the free theory, and need to be altered in the interacting theory.

4.3 Interacting Dynamical Supersymmetries

Just as in the free case, it suffices to determine the form of the dynamical supersymmetry transformations. We write

$$\delta_{\varepsilon\overline{Q}} \varphi^a = \delta_{\varepsilon\overline{Q}}^{free} \varphi^a + \delta_{\varepsilon\overline{Q}}^{int} \varphi^a, \quad \delta_{\varepsilon Q} \varphi^a = \delta_{\varepsilon Q}^{free} \varphi^a + \delta_{\varepsilon Q}^{int} \varphi^a. \quad (30)$$

The expressions $\delta_{\varepsilon\overline{Q}}^{int} \varphi^a$ and $\delta_{\varepsilon Q}^{int} \varphi^a$ are highly restricted, by the following ten constraints:

1. Chirality

$$d^m \delta_{\varepsilon\overline{Q}}^{int} \varphi^a = d^m \delta_{\varepsilon Q}^{int} \varphi^a = 0. \quad (31)$$

2. Both $\delta_{\varepsilon\overline{Q}}^{int} \varphi^a$ and $\delta_{\overline{\varepsilon}Q}^{int} \varphi^a$ are cubic in the superfields.

In three dimensions, canonical Bose fields have mass dimension of one-half, so that the chiral superfield has half-odd integer canonical dimension itself, assuming integer power of derivatives. Since we are looking for a conformal theory with no dimensionful parameters, $\delta_{\varepsilon\overline{Q}} \varphi^a$ and $\delta_{\overline{\varepsilon}Q} \varphi^a$ must then both be an odd power of superfields. Also, conformal invariance requires a Hamiltonian with a local sixth-order interaction in the superfields: the non-linear part of the dynamical supersymmetry transformation must be *cubic* in the superfields[9]: the theory must have a tensor with four indices².

3. Both are independent of x^- , using

$$[\delta_{P+}, \delta_{\varepsilon\overline{Q}}] \varphi^a = [\delta_{P+}, \delta_{\overline{\varepsilon}Q}] \varphi^a = 0. \quad (32)$$

4. $\delta_{\varepsilon\overline{Q}}^{int} \varphi^a$ is independent of x , since

$$[\delta_P, \delta_{\varepsilon\overline{Q}}] \varphi^a = 0. \quad (33)$$

5. Neither have transverse derivatives ∂ : from

$$[\delta_{J+}, \delta_{\varepsilon\overline{Q}}] \varphi^a = -\frac{i}{2} \delta_{\varepsilon\overline{q}} \varphi^a,$$

it follows that

$$[\delta_{J+}, \delta_{\varepsilon\overline{Q}}^{int}] \varphi^a = 0. \quad (34)$$

6. From

$$[\delta_{\overline{\varepsilon}q}, \delta_{\varepsilon\overline{Q}}] \varphi^a = -\overline{\varepsilon}_m \varepsilon^m \delta_P \varphi^a, \quad [\delta_{\varepsilon\overline{q}}, \delta_{\overline{\varepsilon}Q}] \varphi^a = \overline{\varepsilon}_m \varepsilon^m \delta_P \varphi^a,$$

we deduce that

$$[\delta_{\overline{\varepsilon}q}, \delta_{\varepsilon\overline{Q}}^{int}] \varphi^a = [\delta_{\varepsilon\overline{q}}, \delta_{\overline{\varepsilon}Q}^{int}] \varphi^a = 0. \quad (35)$$

7. Proper transformation under J^{+-}

$$[\delta_{J+-}, \delta_{\varepsilon\overline{Q}}^{int}] \varphi^a = -\frac{i}{2} \delta_{\varepsilon\overline{Q}}^{int} \varphi^a, \quad [\delta_{J+-}, \delta_{\overline{\varepsilon}Q}^{int}] \varphi^a = -\frac{i}{2} \delta_{\overline{\varepsilon}Q}^{int} \varphi^a \quad (36)$$

²In $d = 4$, similar considerations suggested a tensor with three indices, f^{abc} , which turned out to be the structure functions of the gauge algebra.

8. Dimension analysis requires

$$[\delta_D, \delta_{\varepsilon\overline{\mathcal{Q}}}^{int}] \varphi^a = \frac{i}{2} \delta_{\varepsilon\overline{\mathcal{Q}}}^{int} \varphi^a, \quad [\delta_D, \delta_{\overline{\varepsilon}\mathcal{Q}}^{int}] \varphi^a = \frac{i}{2} \delta_{\overline{\varepsilon}\mathcal{Q}}^{int} \varphi^a. \quad (37)$$

9. They have opposite $U(1)$ R -charge,

$$[\delta_J, \delta_{\varepsilon\overline{\mathcal{Q}}}^{int}] \varphi^a = \frac{1}{2} \delta_{\varepsilon\overline{\mathcal{Q}}}^{int} \varphi^a, \quad [\delta_J, \delta_{\overline{\varepsilon}\mathcal{Q}}^{int}] \varphi^a = -\frac{1}{2} \delta_{\overline{\varepsilon}\mathcal{Q}}^{int} \varphi^a. \quad (38)$$

10. The eight interacting supersymmetries must also transform as an $SO(8)$ vector: with $\varepsilon'_m = \omega_{mn} \varepsilon^n$,

$$[\delta_{coset}, \delta_{\varepsilon'\overline{\mathcal{Q}}}] \varphi^a = \delta_{\varepsilon'\overline{\mathcal{Q}}} \varphi^a, \quad (39)$$

$$[\delta_{\overline{coset}}, \delta_{\varepsilon\overline{\mathcal{Q}}}] \varphi^a = 0. \quad (40)$$

Similarly, with $\varepsilon'^m = \omega^{mn} \overline{\varepsilon}_n$,

$$[\delta_{\overline{coset}}, \delta_{\overline{\varepsilon}\mathcal{Q}}] \varphi^a = \delta_{\overline{\varepsilon}\mathcal{Q}} \varphi^a, \quad (41)$$

$$[\delta_{coset}, \delta_{\overline{\varepsilon}\mathcal{Q}}] \varphi^a = 0. \quad (42)$$

These ten requirements limit the possible forms of the dynamical supersymmetries.

5 Solving the Kinematical Restrictions

In order to satisfy the first two requirements, we must construct chiral cubic polynomials in the superfields, which requires a bit of algebraic technology.

5.1 Chiral Engineering

Introduce the coherent state operators

$$E_\eta = e^{\eta \cdot \widehat{d}}, \quad (43)$$

where the hat denotes division by ∂^+ , and η^m are arbitrary Grassmann parameters. Since

$$d^m (E_\eta \varphi^a) = i\sqrt{2} \eta^m (E_\eta \varphi^a), \quad (44)$$

$E_\eta \varphi^a$ are eigenstates of the chiral derivatives. It follows that the quadratic combination

$$Z^{bc(\eta)} = (E_\eta \partial^{+B} \varphi^b) (E_{-\eta} \partial^{+C} \varphi^c) , \quad (45)$$

is manifestly chiral,

$$d^m Z^{bc(\eta)} = 0 . \quad (46)$$

Chiral cubic polynomials in the superfields are then constructed in *nested form*,

$$\mathcal{C}^{bcd(\eta, \zeta)} = (E_\eta \partial^{+B} \varphi^b) E_{-\eta} \frac{1}{\partial^{+M}} ((E_\zeta \partial^{+C} \varphi^c) (E_{-\zeta} \partial^{+D} \varphi^d)) , \quad (47)$$

which is manifestly chiral

$$d^m \mathcal{C}^{bcd(\eta, \zeta)} = 0 , \quad (48)$$

and serves as a generating function where the chiral cubic polynomials in the superfields appear as the coefficients in the series expansion in the independent Grassmann variables η and ζ .

5.2 Dynamical Supersymmetry

To find it, we introduce the supersymmetry parameters in the nested Ansatz through the combinations

$$E_\varepsilon = e^{\varepsilon \cdot \widehat{q}} , \quad E_{\bar{\varepsilon}} = e^{\bar{\varepsilon} \cdot \widehat{q}} , \quad (49)$$

which allows us to keep track of requirement (6), without affecting chirality. This leads to the nested ansätze of the form

$$\begin{aligned} \delta_{\varepsilon \overline{Q}} \varphi^a &= \frac{f^{abcd}}{\partial^{+A_\alpha}} \left((E_\varepsilon E_\eta \partial^{+B_\alpha} \varphi^b) E_{-\varepsilon} E_{-\eta} \frac{1}{\partial^{+M_\alpha}} ((E_\zeta \partial^{+C_\alpha} \varphi^c) (E_{-\zeta} \partial^{+D_\alpha} \varphi^d)) \right) , \\ &\equiv \mathcal{K}_\alpha^{a(\varepsilon, \eta, \zeta)} , \end{aligned} \quad (50)$$

keeping only the first order in the supersymmetry parameters ε^m . The f^{abcd} are unknown coefficients, and the exponents A_α , B_α , M_α , C_α , D_α have yet to be determined. In this form, many of the ten requirements are manifestly satisfied:

- Chirality is manifest since the \bar{q}_n anticommute with the chiral derivatives.
- Requirements (3), (4), (5), and (6) are clearly satisfied.
- The proper transformation under J^{+-} , (7), restricts the power of the ∂^+ derivatives so that

$$A_\alpha + M_\alpha - B_\alpha - C_\alpha - D_\alpha + 4 = 0 , \quad (51)$$

which also satisfies the dimension requirement (8).

- The correct $U(1)$ R -charge, requirement (9), demands after some computation

$$\left(\eta^m \frac{\partial}{\partial \eta^m} + \zeta^m \frac{\partial}{\partial \zeta^m} - 4 \right) \mathcal{K}_\alpha^{a(\epsilon, \eta, \zeta)} = 0. \quad (52)$$

- The tenth requirement, that the eight supersymmetries transform as an $SO(8)$ vector, is the hardest to satisfy. Computation of the commutator yields

$$\begin{aligned} [\delta_{coset}, \delta_{\epsilon' \mathcal{Q}}] \varphi^a &= \delta_{\epsilon' \mathcal{Q}} \varphi^a \\ &+ \omega_{mn} \left(\eta^m \eta^n (\widehat{U}_1 + \widehat{U}_2) + \zeta^m \zeta^n (\widehat{U}_3 + \widehat{U}_4) \right) \mathcal{K}_\alpha^{a(\epsilon, \eta, \zeta)}. \end{aligned} \quad (53)$$

Here, \widehat{U}_i means insertion of $\frac{1}{\partial^\mp}$ in the i^{th} position; for instance

$$\begin{aligned} \widehat{U}_2 \mathcal{K}_\alpha^{a(0, \eta, \zeta)} &= \\ \frac{f^{abcd}}{\partial^{+A_\alpha}} \left((E_\eta \partial^{+B_\alpha} \varphi^b) E_{-\eta} \frac{1}{\partial^{+(M_\alpha+1)}} ((E_\zeta \partial^{+C_\alpha} \varphi^c) (E_{-\zeta} \partial^{+D_\alpha} \varphi^d)) \right), \end{aligned}$$

and so on. Hence the tenth kinematical requirement of $SO(8)$ covariance is achieved as long as

$$\left(\eta^m \eta^n (\widehat{U}_1 + \widehat{U}_2) + \zeta^m \zeta^n (\widehat{U}_3 + \widehat{U}_4) \right) \mathcal{K}_\alpha^{a(\epsilon, \eta, \zeta)} = 0. \quad (54)$$

After some algebra, we find two solutions to this equation.

The *odd solution*

$$\delta_{\epsilon' \mathcal{Q}}^{\text{int odd}} \varphi^a = \sum_{\text{odd}} \mathcal{K}_\alpha^{a(\epsilon, \eta, \zeta)} \Big|_{\eta=\zeta=0}, \quad (55)$$

where the sum stands for

$$\sum_{\text{odd}} \equiv \sum_{\alpha=\pm \frac{1}{2}} (-1)^{\alpha+\frac{1}{2}} \frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}}, \quad (56)$$

with

$$\frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} \equiv \frac{\epsilon^{i_1 \dots i_{2-2\alpha} \dots i_4}}{(2+2\alpha)!(2-2\alpha)!} \frac{\partial}{\partial \eta^{i_1 \dots i_{2-2\alpha}}} \frac{\partial}{\partial \zeta^{i_{3-2\alpha} \dots i_4}}. \quad (57)$$

The second is the *even solution*

$$\delta_{\varepsilon\overline{\mathcal{Q}}}^{\text{int even}} \varphi^a = \sum_{\text{even}} \mathcal{K}_{\alpha}^{a(\varepsilon, \eta, \zeta)} \Big|_{\eta=\zeta=0} , \quad (58)$$

with

$$\sum_{\text{even}} \equiv \sum_{\alpha=0, \pm 1} (-1)^{\alpha} \frac{\partial}{\partial \eta^{[2-2\alpha]}} \frac{\partial}{\partial \zeta^{[2+2\alpha]}} . \quad (59)$$

In both cases, the powers of the ∂^+ derivatives are related by

$$A_{\alpha-1} = A_{\alpha} + 1 , \quad B_{\alpha-1} = B_{\alpha} + 1 , \quad M_{\alpha-1} = M_{\alpha} - 2 , \quad (60)$$

as well as

$$C_{\alpha-1} = C_{\alpha} - 1 , \quad D_{\alpha-1} = D_{\alpha} - 1 . \quad (61)$$

Both even and odd solutions are seen to satisfy Eq.(40). Their forms suggest that $SO(8)$ triality is at work, with α denoting the $U(1)$ charges in its vector and spinor representations.

Both solutions are conveniently written in the form

$$\delta_{\varepsilon\overline{\mathcal{Q}}}^{\text{int odd(even)}} \varphi^a \equiv [A_{\alpha}, B_{\alpha}, M_{\alpha}, C_{\alpha}, D_{\alpha}]_{\text{odd(even)}} , \quad (62)$$

with $\alpha = -1/2(-1)$ in the odd(even) case.

It can be checked that these two solutions satisfy the correct commutations with the kinematical conformal supersymmetries

$$[\delta_{\alpha\bar{s}}, \delta_{\varepsilon\overline{\mathcal{Q}}}] \varphi^a = \left(i\delta_D \varphi^a - i\delta_{J^{+-}} \varphi^a + \frac{1}{2} \delta_J \varphi^a \right) + \frac{1}{\sqrt{2}} \delta_{SO(6)} \varphi^a , \quad (63)$$

as well as

$$[\delta_{\alpha\bar{s}}, \delta_{\varepsilon\overline{\mathcal{Q}}}] \varphi^a = \delta_{\text{coset}} \varphi^a . \quad (64)$$

Finally, we note that the conjugate supersymmetries are obtained by simply changing E_{ε} into $E_{\bar{\varepsilon}}$.

5.3 Hamiltonian and Boost

In the previous section, the form of the dynamical supersymmetry transformations have been narrowed down to two solutions with yet undetermined powers of the light-cone derivatives. In the $d = 4$ SuperYang-Mills case, their values were determined from the vanishing of the commutator between the light-cone boost and Hamiltonian. We expect the same to hold in the $d = 3$ theory.

The light-cone Hamiltonian is computed from

$$[\delta_{\bar{\varepsilon}\mathcal{Q}}^{free} + \delta_{\bar{\varepsilon}\mathcal{Q}}^{int}, \delta_{\varepsilon\bar{\mathcal{Q}}}^{free} + \delta_{\varepsilon\bar{\mathcal{Q}}}^{int}] \varphi^a = \sqrt{2} \bar{\varepsilon}_m \varepsilon^m \delta_{\mathcal{P}^-} \varphi^a. \quad (65)$$

The commutator yields terms linear and quadratic in f^{abcd} . The first order stems from

$$[\delta_{\bar{\varepsilon}\mathcal{Q}}^{free}, \delta_{\varepsilon\bar{\mathcal{Q}}}^{int}] \varphi^a + [\delta_{\bar{\varepsilon}\mathcal{Q}}^{int}, \delta_{\varepsilon\bar{\mathcal{Q}}}^{free}] \varphi^a \quad (66)$$

The results of the computation are expressed in terms of

$$\begin{aligned} K_\alpha^{a[r,1]} &\equiv (E_r U_1)(E_{-r} U_2) \mathcal{K}_\alpha^{a(0,\eta,\zeta)}, \\ &= \frac{f^{abcd}}{\partial^+ A_\alpha} \left((E_r E_\eta \partial^{+B_\alpha} \varphi^b) E_{-r} E_{-\eta} \frac{1}{\partial^+ M_\alpha} ((E_\zeta \partial^{+C_\alpha} \varphi^c)(E_{-\zeta} \partial^{+D_\alpha} \varphi^d)) \right), \end{aligned}$$

and

$$\begin{aligned} K_\alpha^{a[1,r]} &\equiv (E_r U_3)(E_{-r} U_4) \mathcal{K}_\alpha^{a(0,\eta,\zeta)}, \\ &= \frac{f^{abcd}}{\partial^+ A_\alpha} \left((E_\eta \partial^{+B_\alpha} \varphi^b) E_{-\eta} \frac{1}{\partial^+ M_\alpha} ((E_r E_\zeta \partial^{+C_\alpha} \varphi^c)(E_{-r} E_{-\zeta} \partial^{+D_\alpha} \varphi^d)) \right), \end{aligned}$$

where the transverse derivative is introduced through

$$E_r = e^{r\hat{\partial}}, \quad (67)$$

and r is a dimensionless parameter.

The computation of these commutators yields, for the odd case,

$$\delta_{\mathcal{P}^-}^{int odd} \varphi^a = \frac{\partial}{\partial r} \left(\sum_{even} K_\alpha^{a[1,r]} + \sum_{odd} K_{\alpha+\frac{1}{2}}^{a[r,1]} \right)_{r=0}, \quad (68)$$

for the linear part in f^{abcd} .

A similar expression is found in the even case,

$$\delta_{\mathcal{P}^-}^{int even} \varphi^a = \frac{\partial}{\partial r} \left(\sum_{odd} K_\alpha^{a[1,r]} - \sum_{even} K_{\alpha+\frac{1}{2}}^{a[r,1]} \right)_{r=0}. \quad (69)$$

The boost transformation is computed from the commutator

$$\delta_{\mathcal{J}^-} \varphi^a = -\frac{i}{2} [\delta_K, \delta_{\mathcal{P}^-}] \varphi^a. \quad (70)$$

for both odd and even cases. Its expression is not particularly enlightening, and will be published elsewhere[6].

5.4 Dynamical Constraints

The next step is to require

$$[\delta_{\mathcal{P}^-}, \delta_{\mathcal{J}^-}] \varphi^a = 0. \quad (71)$$

This condition, as in the Yang-Mills case, is expected to fix the unknown exponents, and the interactions. After a lengthy calculation, keeping only the terms linear in f^{abcd} , the result can be written in the form

$$[\delta_{\mathcal{P}^-}^{odd}, \delta_{\mathcal{J}^-}^{odd}] \varphi^a = \mathcal{S} \frac{\partial^2}{\partial r \partial r'} (F \mathcal{O}_1^{odd} + G \mathcal{O}_2^{odd})_{r=r'=0}, \quad (72)$$

where \mathcal{S} is a shift operator

$$\mathcal{S}: \quad A_\alpha \rightarrow A_\alpha + 1, \quad B_\alpha \rightarrow B_\alpha + 1, \quad M_\alpha \rightarrow M_\alpha - 1. \quad (73)$$

In addition,

$$F \equiv (B_{-\frac{1}{2}} - 3) \hat{U}_1 + (M_{-\frac{1}{2}} - C_{-\frac{1}{2}} - D_{-\frac{1}{2}} + 3) \hat{U}_2, \quad (74)$$

$$G \equiv 2(C_{-\frac{1}{2}} - \frac{3}{2}) \hat{U}_3 - 2(D_{-\frac{1}{2}} - \frac{3}{2}) \hat{U}_4, \quad (75)$$

and

$$\mathcal{O}_1^{odd} = \sum_{odd} (K_{\alpha}^{[rr',1]} - K_{\alpha+1}^{[rr',1]}) + 2 \sum_{even} K_{\alpha+\frac{1}{2}}^{[r,r']}, \quad (76)$$

$$\mathcal{O}_2^{odd} = \sum_{odd} K_{\alpha+1}^{[r,r']} - \frac{1}{2} \sum_{even} (K_{\alpha+\frac{1}{2}}^{[r,r']} - K_{\alpha+\frac{3}{2}}^{[1,rr']}). \quad (77)$$

A similar equation obtains in the even case.

We can envisage two types of solutions to the vanishing of this commutator.

- The “trivial” solution is when $F = G = 0$, which determines the values of all the exponents, and leads to

$$\delta_{\varepsilon \mathcal{Q}}^{\text{int } odd} \varphi^a = [2, 3, 0, \frac{3}{2}, \frac{3}{2}]_{odd}, \quad (78)$$

which exists only if $f^{abcd} = -f^{abdc}$.

The even solution, given by,

$$\delta_{\varepsilon \mathcal{Q}}^{\text{int } even} \varphi^a = [\frac{5}{2}, \frac{7}{2}, -1, 1, 1]_{even}, \quad (79)$$

requires $f^{abcd} = +f^{abdc}$.

In both cases, there are no further symmetry requirements on f^{abcd} . Both solutions require fractional powers of ∂^+ , which have to be further interpreted.

We have not checked the validity of this solution any further: there remains to check the vanishing of the commutators

$$[\delta_K, \delta_{\mathcal{K}-}] \varphi^a, \quad [\delta_{\mathcal{P}-}, \delta_{\mathcal{K}-}] \varphi^a, \quad [\delta_{\mathcal{J}-}, \delta_{\mathcal{K}-}] \varphi^a. \quad (80)$$

In the Yang-Mills case, these did not put any further restrictions on the solution. The vanishing of the second one may be explained by the Jacobi identity:

$$[\delta_{\mathcal{P}-}, [\delta_K, \delta_{\mathcal{J}-}]] + [\delta_K, [\delta_{\mathcal{J}-}, \delta_{\mathcal{P}-}]] + [\delta_{\mathcal{J}-}, [\delta_{\mathcal{P}-}, \delta_K]] = 0. \quad (81)$$

Since

$$[\delta_K, \delta_{\mathcal{J}-}] \varphi^a = -i \delta_{\mathcal{K}-} \varphi^a, \quad (82)$$

it follows that

$$[\delta_{\mathcal{P}-}, \delta_{\mathcal{J}-}] \varphi^a = 0. \quad (83)$$

The algebraic validity of the fractional power solution hinges on the first commutator

$$[\delta_K, \delta_{\mathcal{K}-}] \varphi^a = 0, \quad (84)$$

which we have not yet checked. Through the Jacobi identity, it would ensure that

$$[\delta_{\mathcal{J}-}, \delta_{\mathcal{K}-}] \varphi^a = 0. \quad (85)$$

- The less trivial solution(s) relies on the symmetries of f^{abcd} under the interchange of *three* of its indices. It appears to lead uniquely to the BLG solution; since at the time of Shifmania, we had not obtained it, its details will appear elsewhere[6].

Much remains to be done. For one, we have not derived the quadratic term in f^{abcd} in the Hamiltonian. In the Yang-Mills case, this led to the Jacobi identity of the f^{abc} and identified them as structure functions.

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References

- [1] Jonathan Bagger, and Neil Lambert, *Phys. Rev.*, **D75** 045020 (2007); *Phys. Rev.*, **D77** 065008 (2008)
- [2] Andreas Gustavsson, *Nucl. Phys.* **B811**, 66 (2009).
- [3] Bengt E. W. Nilsson, *Class. Quant. Grav* **26**, 175001 (2009).
- [4] Sudarshan Ananth, Lars Brink, Sung-Soo Kim, and Pierre Ramond, *Nucl. Phys.* **B722**, 166 (2005).
- [5] Lars Brink, John H. Schwarz, and Joël Scherk, *Nucl. Phys.* **B121**, 77 (1977); Ferdinando Gliozzi, Joël Scherk, and David I. Olive, *Nucl. Phys.* **B122**, 253 (1977).
- [6] D. Belyaev, L. Brink, S-S Kim, and P. Ramond, in preparation.
- [7] Werner Nahm, *Nucl. Phys.* **B135**, 149 (1978).
- [8] Edward Witten, “*Some Comments on String Dynamics*”, in Los Angeles 1995, Future perspectives in string theory; Michael R. Douglas, and Gregory W. Moore, “*D-branes, quivers, and ALE instantons*”, (1996) hep-th/9603167; Jacques Distler, and Amihay Hanany, *Nucl. Phys.* **B490**, 75(1997); Juan Martin Maldacena, *Adv.Theor. Math. Phys.* **2** 231 (1998), and *Int. J.Theor. Phys.* **38** 1113 (1999) Ofer Aharony, Amihay Hanany, and Barak Kol, *JHEP* **9801** 002(1998); Christophe Grojean, and Jihad Mourad, *Class. Quant. Grav.* **15**, 3397 (1998); Par Arvidsson, Erik Flink, and Måns Henningson, *JHEP* **0405**, 048 (2004); John H. Schwarz, *JHEP* **0411**, 078 (2004).
- [9] Anirban Basu and Jeffrey A. Harvey, *Nucl. Phys.* **B713**,136 (2005).